On the Fibonacci and Lucas numbers, their sums and permanents of one type of Hessenberg matrices

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Abstract

At this paper, we derive some relationships between permanents of one type of lower-Hessenberg matrix and the Fibonacci and Lucas numbers and their sums.

1 Introduction

The well-known Fibonacci and Lucas sequences are recursively defined by

$$F_{n+1} = F_n + F_{n-1}, \ n \geq 1$$
$$L_{n+1} = L_n + L_{n-1}, \ n \geq 1$$

with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$. The first few values of the sequences are given below:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

The permanent of a matrix is similar to the determinant but all of the signs used in the Laplace expansion of minors are positive. The permanent of an n-square matrix is defined by

$$per A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n [1].

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Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A is *contractible* on column k, if column k contains exactly two non zero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{k:ij} = [A_{ij:k}^T]^T$ is called the contraction of A on row k relative to columns i and j. We know that if A is a nonnegative matrix and B is a contraction of A [2], then

$$per A = per B. \tag{1}$$

It is known that there are a lot of relations between determinants or permanents of matrices and well-known number sequences. For example, the authors [2] investigate relationships between permanents of one type of Hessenberg matrix the Pell and Perrin numbers.

In [3], Lee defined the matrix

$$\pounds_n = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & & \vdots \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and showed that

$$per(\pounds_n) = L_{n-1}$$

where L_n is the nth Lucas number.

In [4], the author investigate general tridiagonal matrix determinants and permanents. Also he showed that the permanent of the tridiagonal matrix based on $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}$, $\{b_i\}$, $\{c_i\}$.

In [5], the authors give $(0, 1, -1)$ tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Also, they give an $n \times n$ $(-1, 1)$ matrix S,

$$S = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix}.$$

such that per$A$=det$(A \circ S)$, where $A \circ S$ denotes Hadamard product of A and S.

In the present paper, we consider a particular case of lower Hessenberg matrices. Then, we show that the permanents of these type of matrices are related with Fibonacci and Lucas numbers and their sums.

2 Determinantal representation of Fibonacci and Lucas numbers and their sums

In this section, we define one type of lower Hessenberg matrix and show that the permanents of these type of matrices are Fibonacci, Lucas numbers and their sums.

Let $H_n = [h_{ij}]_{n \times n}$ be an n-square lower Hessenberg matrix in which the superdiagonal entries are alternating -1s and 1s, starting with -1, the main diagonal entries are 2s, except the last one which is 1, the subdiagonal entries are 0s, the lower-subdiagonal entries are 1s and otherwise 0. That is:

$$
H_n = \begin{bmatrix}
2 & -1 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & \ddots & & \\
& & & & 1 & 0 & 2 & (-1)^{n-1} \\
& & & & & 1 & 0 & 1
\end{bmatrix}
\tag{2}
$$

Theorem 1 *Let H_n be as in (2), then*

$$
per H_n = per H_n^{(n-2)} = F_{n+1}
$$

where F_n is the nth Fibonacci number.

Proof. By definition of the matrix H_n, it can be contracted on column n. Let $H_n^{(r)}$ be the rth contraction of H_n. If $r = 1$, then

$$
H_n^{(1)} = \begin{bmatrix}
2 & -1 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & & \ddots & \\
& & & & 1 & 0 & 2 & (-1)^{n-2} \\
& & & & & 1 & (-1)^{n-1} & 2
\end{bmatrix}.
$$

Since $H_n^{(1)}$ also can be contracted according to the last column,

$$
H_n^{(2)} = \begin{bmatrix}
2 & -1 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & & \ddots & \\
& & & & 1 & 0 & 2 & (-1)^{n-3} \\
& & & & & 2 & (-1)^{n-2} & 3
\end{bmatrix}.
$$

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Furthermore, the matrix $H_n^{(2)}$ can be contracted on the last column, that is

$$
H_n^{(3)} = \begin{bmatrix}
2 & -1 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & & \ddots & \\
& & & 1 & 0 & 2 & & (-1)^{n-4} \\
& & & & 3 & (-1)^{n-3}2 & 5 &
\end{bmatrix}.
$$

Continuing this method, we obtain the rth contraction

$$
H_n^{(r)} = \begin{bmatrix}
2 & -1 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
& 1 & 0 & 2 & 1 & & \\
& & \ddots & \ddots & \ddots & & \ddots \\
& & & 1 & 0 & 2 & (-1)^{r-1} \\
& & & & F_{r+1} & (-1)^r(F_{r+2}-F_{r+1}) & F_{r+2}
\end{bmatrix}, \; n \text{ is even}
$$

$$
H_n^{(r)} = \begin{bmatrix}
2 & -1 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
& 1 & 0 & 2 & 1 & & \\
& & \ddots & \ddots & \ddots & & \ddots \\
& & & 1 & 0 & 2 & (-1)^r \\
& & & & F_{r+1} & (-1)^{r-1}(F_{r+2}-F_{r+1}) & F_{r+2}
\end{bmatrix}, \; n \text{ is odd}
$$

where $2 \leq r \leq n-4$. Hence

$$
H_n^{(n-3)} = \begin{bmatrix}
2 & -1 & 0 \\
0 & 2 & 1 \\
F_{n-2} & (F_{n-2}-F_{n-1}) & F_{n-1}
\end{bmatrix}
$$

which by contraction of $H_n^{(n-3)}$ on column 3,

$$
H_n^{(n-2)} = \begin{bmatrix}
2 & -1 \\
F_{n-2} & F_n
\end{bmatrix}.
$$

By (1), we have $per H_n = per H_n^{(n-2)} = F_{n+1}$. ∎

Let $K_n = [k_{ij}]_{n \times n}$ be an n-square lower Hessenberg matrix in which the superdiagonal entries are alternating -1s and 1s starting with 1, except the first one which is -3, the main diagonal entries are 2s, except the last one which is 1, the subdiagonal entries are 0s, the lower-subdiagonal entries are 1s and otherwise 0. Clearly:

$$K_n = \begin{bmatrix} 2 & -3 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & 2 & (-1)^{n-1} \\ & & & & 1 & 0 & 1 \end{bmatrix} \tag{3}$$

Theorem 2 *Let K_n be as in (3), then*

$$per K_n = per K_n^{(n-2)} = L_{n-2}$$

where L_n is the nth Lucas number.

Proof. By definition of the matrix K_n, it can be contracted on column n. That is,

$$K_n^{(1)} = \begin{bmatrix} 2 & -3 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{n-2} \\ & & & & 1 & (-1)^{n-1} & 2 \end{bmatrix}.$$

$K_n^{(1)}$ also can be contracted on the last column,

$$K_n^{(2)} = \begin{bmatrix} 2 & -3 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{n-3} \\ & & & & 2 & (-1)^{n-2} & 3 \end{bmatrix}.$$

$K_n^{(2)}$ also can be contracted on the last column,

$$K_n^{(3)} = \begin{bmatrix} 2 & -3 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{n-4} \\ & & & & 3 & 2(-1)^{n-3} & 5 \end{bmatrix}.$$

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Going with this process, we have

$$
K_n^{(r)} = \begin{bmatrix}
2 & -3 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
& 1 & 0 & 2 & 1 & & \\
& & \ddots & \ddots & \ddots & & \ddots \\
& & & 1 & 0 & 2 & (-1)^{r-1} \\
& & & & F_{r+1} & (-1)^{r-2}(F_{r+2} - F_{r+1}) & F_{r+2}
\end{bmatrix}, \ n \text{ is even}
$$

$$
K_n^{(r)} = \begin{bmatrix}
2 & -3 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
& 1 & 0 & 2 & 1 & & \\
& & \ddots & \ddots & \ddots & & \ddots \\
& & & 1 & 0 & 2 & (-1)^{r} \\
& & & & F_{r+1} & (-1)^{r-1}(F_{r+2} - F_{r+1}) & F_{r+2}
\end{bmatrix}, \ n \text{ is odd}
$$

for $2 \le r \le n - 4$. Hence

$$
K_n^{(n-3)} = \begin{bmatrix}
2 & -3 & 0 \\
0 & 2 & 1 \\
F_{n-3} & F_{n-3} - F_{n-1} & F_{n-1}
\end{bmatrix}
$$

which by contraction of $K_n^{(n-3)}$ on column 3, gives

$$
K_n^{(n-2)} = \begin{bmatrix}
2 & -3 \\
F_{n-2} & F_n
\end{bmatrix}.
$$

By applying (1), we have $per K_n = per K_n^{(n-2)} = 2F_n - 3F_{n-2} = L_{n-2}$, which is desired. ∎

Let $M_n = [m_{ij}]_{n \times n}$ be an n-square lower Hessenberg matrix in which the superdiagonal entries are alternating -1s and 1s, starting with -1, the main diagonal entries are 2s, the subdiagonal entries are 0s, the lower-subdiagonal entries are 1s and otherwise 0. In other words:

$$
M_n = \begin{bmatrix}
2 & -1 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
& 1 & 0 & 2 & 1 & & \\
& & \ddots & \ddots & \ddots & \ddots & \\
& & & 1 & 0 & 2 & (-1)^{n-1} \\
& & & & 1 & 0 & 2
\end{bmatrix}
\tag{4}
$$

Theorem 3 *Let M_n be as in (4), then*

$$
per M_n = per M_n^{(n-2)} = \sum_{i=0}^{n-1} F_i
$$

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where F_n is the nth Fibonacci number.

Proof. By definition of the matrix M_n, it can be contracted on column n. That is,

$$M_n^{(1)} = \begin{bmatrix} 2 & -1 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & 1 & 0 & 2 & & (-1)^{n-2} \\ & & & & 2 & (-1)^{n-1} & & 4 \end{bmatrix}.$$

$M_n^{(1)}$ also can be contracted on the last column,

$$M_n^{(2)} = \begin{bmatrix} 2 & -1 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & 1 & 0 & 2 & & (-1)^{n-3} \\ & & & & 4 & 2(-1)^{n-2} & & 7 \end{bmatrix}.$$

$M_n^{(2)}$ also can be contracted on the last column,

$$M_n^{(3)} = \begin{bmatrix} 2 & -1 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & 1 & 0 & 2 & & (-1)^{n-4} \\ & & & & 7 & 4(-1)^{n-3} & & 12 \end{bmatrix}.$$

Going with this process, we have

$$
M_n^{(r)} = \begin{bmatrix}
2 & -1 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & & \ddots & \\
& & & 1 & 0 & & 2 & (-1)^r \\
& & & & \displaystyle\sum_{i=0}^{r+1} F_i & (-1)^{r-1}\displaystyle\sum_{i=0}^{r} F_i & \displaystyle\sum_{i=0}^{r+2} F_i
\end{bmatrix} , \; n \text{ is odd}
$$

$$
M_n^{(r)} = \begin{bmatrix}
2 & -1 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & & \ddots & \\
& & & 1 & 0 & & 2 & (-1)^{r-1} \\
& & & & \displaystyle\sum_{i=0}^{r+1} F_i & (-1)^{r-2}\displaystyle\sum_{i=0}^{r} F_i & \displaystyle\sum_{i=0}^{r+2} F_i
\end{bmatrix} , \; n \text{ is even}
$$

for $2 \leq r \leq n-4$. Hence

$$
M_n^{(n-3)} = \begin{bmatrix}
2 & -1 & 0 \\
0 & 2 & 1 \\
\displaystyle\sum_{i=0}^{n-2} F_i & -\displaystyle\sum_{i=0}^{n-3} F_i & \displaystyle\sum_{i=0}^{n-1} F_i
\end{bmatrix}
$$

which by contraction of $M_n^{(n-3)}$ on column 3, gives

$$
M_n^{(n-2)} = \begin{bmatrix}
2 & -1 \\
\displaystyle\sum_{i=0}^{n-4} F_i & \displaystyle\sum_{i=0}^{n-2} F_i
\end{bmatrix} .
$$

By applying (1), we have

$$
per M_n = per M_n^{(n-2)} = \sum_{i=0}^{n-1} F_i
$$

which is desired. ∎

Let $N_n = [n_{ij}]_{n \times n}$ be an n-square lower Hessenberg matrix in which the superdiagonal entries are alternating -1s and 1s starting with 1, except the first one which is -2, the main diagonal entries are 2s, except the first one is 3, the subdiagonal entries are 0s, the lower-subdiagonal entries are 1s and otherwise 0. In this content:

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$$N_n = \begin{bmatrix} 3 & -2 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 1 & 0 & 2 & (-1)^{n-1} \\ & & & & & 1 & 0 & 2 \end{bmatrix} \qquad (5)$$

Theorem 4 *Let N_n be an n-square matrix $(n \geq 2)$ as in (5), then*

$$per N_n = per N_n^{(n-2)} = \sum_{i=0}^{n} L_i$$

where L_n is the nth Lucas number.

Proof. By definition of the matrix N_n, it can be contracted on column n. That is,

$$N_n^{(1)} = \begin{bmatrix} 3 & -2 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & & 1 & 0 & 2 & (-1)^{n} \\ & & & & & 2 & (-1)^{n-1} & 4 \end{bmatrix}.$$

$N_n^{(1)}$ also can be contracted on the last column,

$$N_n^{(2)} = \begin{bmatrix} 3 & -2 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & & 1 & 0 & 2 & (-1)^{n-1} \\ & & & & & 4 & 2(-1)^{n-2} & 7 \end{bmatrix}.$$

$N_n^{(2)}$ also can be contracted on the last column,

$$N_n^{(3)} = \begin{bmatrix} 3 & -2 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & & 1 & 0 & 2 & (-1)^{n-2} \\ & & & & & 7 & 4(-1)^{n-3} & 12 \end{bmatrix}.$$

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Going with this process, we have

$$
N_n^{(r)} = \begin{bmatrix}
3 & -2 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & & \ddots & \\
& & & 1 & 0 & 2 & & (-1)^r \\
& & & & \sum_{i=0}^{r+1} F_i & (-1)^{r-1}\sum_{i=0}^{r} F_i & \sum_{i=0}^{r+2} F_i
\end{bmatrix}, \; n \text{ is odd}
$$

$$
N_n^{(r)} = \begin{bmatrix}
3 & -2 & & & & & & \\
0 & 2 & 1 & & & & & \\
1 & 0 & 2 & -1 & & & & \\
& 1 & 0 & 2 & 1 & & & \\
& & \ddots & \ddots & \ddots & & \ddots & \\
& & & 1 & 0 & 2 & & (-1)^{r-1} \\
& & & & \sum_{i=0}^{r+1} F_i & (-1)^{r}\sum_{i=0}^{r} F_i & \sum_{i=0}^{r+2} F_i
\end{bmatrix}, \; n \text{ is even}
$$

for $2 \leq r \leq n-4$. Hence

$$
N_n^{(n-3)} = \begin{bmatrix}
3 & -2 & 0 \\
0 & 2 & 1 \\
\sum_{i=0}^{n-2} F_i & -\sum_{i=0}^{n-3} F_i & \sum_{i=0}^{n-1} F_i
\end{bmatrix}
$$

which by contraction of $N_n^{(n-3)}$ on column 3, gives

$$
N_n^{(n-2)} = \begin{bmatrix}
3 & -2 \\
\sum_{i=0}^{n-2} F_i & \sum_{i=0}^{n} F_i
\end{bmatrix}.
$$

By applying (1), we have

$$
\begin{aligned}
per N_n &= per N_n^{(n-2)} \\
&= F_{n+1} + \sum_{i=0}^{n+1} F_i \\
&= \sum_{i=0}^{n} L_i
\end{aligned}
$$

which is desired. ∎

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